# An asymptotic expansion for a ratio of products of gamma functions

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**Abstract**

An asymptotic expansion of a ratio of products of gamma functions is derived. It generalizes a formula which was stated by Dingle, first proved by Paris, and recently reconsidered by Olver.

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## 1 Introduction

Our starting point is the Gaussian hypergeometric function $F(a, b; c; z)$ and its series representation

$$\frac{1}{\Gamma(c)} F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\Gamma(c+n) n!} z^n, \quad |z| < 1,$$

which here is written in terms of Pochhammer symbols

$$(x)_n = x(x+1) \dots (x+n-1) = \Gamma(x+n)/\Gamma(x).$$

The hypergeometric series appears as one solution of the Gaussian (or hypergeometric) differential equation, which is characterized by its three regular

singular points at $z = 0, 1, \infty$. The local series solutions at 0 and 1 of this differential equation are connected by the continuation formula [1]

$$\frac{1}{\Gamma(c)}F(a,b;c;z) = \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;1+a+b-c;1-z)$$

$$+ \frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b;1+c-a-b;1-z), \qquad (1)$$

$$(|\arg(1-z)| < \pi).$$

Here we want to show that Eq. (1) implies an interesting asymptotic expansion for a ratio of products of gamma functions, of which only a special case was known before.

By applying the method of Darboux [4, 8] to (1), we derive in Sec. 2 the formula in question. The behaviour of this and a related formula is discussed in Sec. 3 and illustrated by a few numerical examples.

# 2 Derivation of an asymptotic expansion for a ratio of products of gamma functions

It is well-known that the late coefficients of a Taylor series expansion contain information about the nearest singular point of the expanded function [3]. In this respect we want to analyze the continuation formula (1), in which then only the second, at $z = 1$ singular term $R$ is relevant, which may be written as

$$R = \frac{\Gamma(a+b-c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{\infty} \frac{(c-a)_m(c-b)_m}{\Gamma(1+c-a-b+m)m!}(1-z)^{c-a-b+m}.$$

By means of the binomial theorem in its hypergeometric-series-form , we may expand the power factor

$$(1-z)^{c-a-b+m} = \sum_{n=0}^{\infty} \frac{\Gamma(a+b-c-m+n)}{\Gamma(a+b-c-m)n!}z^n.$$

Interchanging then the order of the summations and simplifying by means of the reflection formula of the gamma function, we arrive at

$$R = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^m \frac{(c-a)_m(c-b)_m}{m!} \frac{\Gamma(a+b-c-m+n)}{n!}z^n.$$

This is to be compared with the left-hand side $L$ of (1), which is

$$L = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n.$$

Comparison of the coefficients of these two power series, which according to Darboux [4] and Schäfke and Schmidt [8] should agree asymptotically as $n \to \infty$, then yields

$$\begin{aligned} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} &= \sum_{m=0}^{M} (-1)^m \frac{(c-a)_m (c-b)_m}{m!} \Gamma(a+b-c-m+n) \\ &\quad + O(\Gamma(a+b-c-M-1+n)). \end{aligned} \tag{2}$$

By means of

$$O(\Gamma(a+b-c-M-1+n)) = \Gamma(a+b-c+n) O(n^{-M-1})$$

and the reflection formula of the gamma function, the relevant formula (2) may also be written as

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} = 1 + \sum_{m=1}^{M} \frac{(c-a)_m (c-b)_m}{m!(1+c-a-b-n)_m} + O(n^{-M-1}). \tag{3}$$

The asymptotic expansion for a ratio of products of gamma functions in this form (3) or the other (2) seems to be new. It is only the special case when $c = 1$ which is known. This special case was stated by Dingle[2], first proved by Paris[7], and reconsidered recently by Olver[5], who has found a simple direct proof. His proof, as well as the proof of Paris, can be adapted easily to the more general case when $c$ is different from 1 . Still another proof is available [6] which includes an integral representation of the remainder term. Our derivation of Eq. (2) or (3) is significantly different from all the earlier proofs of the case when $c = 1$.

# 3 Discussion and numerical examples

We now want to discuss our result in the form (3). First we observe that the substitution $c \to a+b-c$ leads to the related formula

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} = 1 + \sum_{m=1}^{M} \frac{(a-c)_m (b-c)_m}{m!(1-c-n)_m} + O(n^{-M-1}). \tag{4}$$

Which of (3) or (4) is more advantageous numerically depends on the values of the parameters, and in this respect the two formulas complement each other. Table 1 shows an example with a set of parameters for which (3) gives more accurate values than (4), while Table 2 contains an example for which (4) is superior to (3).

For finite $n$ and $M \to \infty$ the series on the right-hand side of (3) converges if $\mathrm{Re}(1-c-n) > 0$. The same is true for (4) if $\mathrm{Re}(1+c-a-b-n) > 0$ . Then, in both cases, the Gaussian summation formula yields

$$\frac{\Gamma(1-c-n)\Gamma(1+c-a-b-n)}{\Gamma 1-a-n)\Gamma(1-b-n)},$$

which, by means of the reflection formula of the gamma function, is seen to be equal to

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} \frac{\sin(\pi[a+n])\sin(\pi[b+n])}{\sin(\pi[c+n])\sin(\pi[a+b-c+n])}. \tag{5}$$

Otherwise (2) – (4) are divergent asymptotic expansions as $n \to \infty$.

Although in our derivation $n$ is a sufficiently large positive integer, the asymptotic expansions (2) – (4) are expected to be valid in a certain sector of the complex $n$ -plane, and in fact, the proofs of Paris [7] and of Olver [6] apply to complex values of $n$.

If the series in (3) or (4) converge, their sums are equal to (5), which generally (if neither $c-a$ nor $c-b$ is equal to an integer ) is different from the left-hand side of (3) or (4). Therefore (3) and (4) can be valid only in the half-planes in which the series do not converge. This means that (3) is an asymptotic expansion as $n \to \infty$ in the half-plane $\mathrm{Re}(c-1+n) \geq 0$, and (4) is an asymptotic expansion as $n \to \infty$ in the half-plane $\mathrm{Re}(a+b-c-1+n) \geq 0$. Otherwise the series on the right-hand sides represent a different function, namely (5).

A few numerical examples may serve for demonstration of these facts. In Table 3 , the series converge to (5) for $n = 10$ , and therefore (3) and (4) are not valid. For $n = 20$, on the other hand, the series diverge and so (3) and (4) hold. The transition between the two regions is at the line $\mathrm{Re}(n) = 12.4$ in case of (3) or $\mathrm{Re}(n) = 12.5$ in case of (4). In Table 4, we see convergence for $n = -15$ and divergence for $n = -5$, the transition between the two regions being at the line $\mathrm{Re}(n) = -10.4$ in case of (3) or $\mathrm{Re}(n) = -10.5$ in case of (4) .

## References


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| | $M$ | right-hand side of (3) | | right-hand side of(4) | |
|---|---|---|---|---|---|
| $n = 10$ | 1 | 0.9771429 | | 0.9744681 | |
| | 2 | 0.9773113 | | 0.9780243 | |
| | 3 | 0.9772978 | | 0.9769927 | |
| | 4 | 0.9773005 | | 0.9774980 | $\leftarrow$ |
| | 5 | 0.9772995 | $\leftarrow$ | 0.9771117 | $\leftarrow$ |
| | 6 | 0.9773001 | $\leftarrow$ | 0.9775615 | |
| | 7 | 0.9772995 | | 0.9767519 | |
| | 8 | 0.9773003 | | 0.9791530 | |
| | 9 | 0.9772983 | | 0.9652341 | |
| | 10 | 0.9773079 | | 1.2823765 | |
| | exact value of (3) or (4): 0.97729983 | | | | |

Table 1: Values of the right-hand sides of (3) and (4) for the parameters $a = 0.7$, $b = 1.2$, $c = 0.4$.

| | $M$ | right-hand side of (3) | | right-hand side of(4) | |
|---|---|---|---|---|---|
| $n = 10$ | 1 | 0.968000 | | 0.972093 | |
| | 2 | 0.973760 | | 0.972350 | |
| | 3 | 0.971512 | $\leftarrow$ | 0.972324 | |
| | 4 | 0.973078 | $\leftarrow$ | 0.972331 | |
| | 5 | 0.971231 | | 0.972327 | $\leftarrow$ |
| | 6 | 0.975016 | | 0.972330 | $\leftarrow$ |
| | 7 | 0.959571 | | 0.972325 | |
| | 8 | 1.179434 | | 0.972342 | |
| | 9 | 4.748048 | | 0.972163 | |
| | 10 | 26.430946 | | 0.968966 | |
| | exact value of (3) or (4): 0.97232850 | | | | |

Table 2: Values of the right-hand sides of (3) and (4) for the parameters $a = -0.7$, $b = -1.2$, $c = -0.4$.

| | $M$ | right-hand side of (3) | right-hand side of(4) |
|---|---|---|---|
| $n = 10$ | 1 | 0.976000 | 0.975000 |
| | 2 | 0.972434 | 0.971912 |
| | 3 | 0.971341 | 0.971037 |
| | 4 | 0.970882 | 0.970687 |
| | 5 | 0.970651 | 0.970517 |
| | 6 | 0.970520 | 0.970423 |
| | 7 | 0.970440 | 0.970367 |
| | 8 | 0.970388 | 0.970331 |
| | 9 | 0.970352 | 0.970307 |
| | 10 | 0.970326 | 0.970290 |
| | exact value of(3) or (4): 1.94045281 | | |
| | exact value of (5): 0.97022640 $\leftarrow$ | | |
| $n = 20$ | 1 | 1.008000 | 1.007895 |
| | 2 | 1.007360 | 1.007392 |
| | 3 | 1.007521 | 1.007504 |
| | 4 | 1.007438 $\leftarrow$ | 1.007452 $\leftarrow$ |
| | 5 | 1.007515 $\leftarrow$ | 1.007497 $\leftarrow$ |
| | 6 | 1.007385 | 1.007426 |
| | 7 | 1.007839 | 1.007650 |
| | 8 | 1.002201 | 1.005398 |
| | 9 | 0.921096 | 0.965891 |
| | 10 | 0.478588 | 0.740024 |
| | exact value of (3) or (4): 1.00747290 $\leftarrow$ | | |
| | exact value of (5): 0.50373645 | | |

Table 3: Values of the right-hand sides of (3) and (4) for the parameters $a = -11.7$, $b = -11.2$, $c = -11.4$ .

| | $M$ | right-hand side of (3) | | right-hand side of(4) | |
|---|---|---|---|---|---|
| $n=-15$ | 1 | 0.986667 | | 0.986957 | |
| | 2 | 0.985648 | | 0.985745 | |
| | 3 | 0.985453 | | 0.985492 | |
| | 4 | 0.985397 | | 0.985415 | |
| | 5 | 0.985376 | | 0.985386 | |
| | 6 | 0.985368 | | 0.985373 | |
| | 7 | 0.985363 | | 0.985367 | |
| | 8 | 0.985361 | | 0.985363 | |
| | 9 | 0.985360 | | 0.985361 | |
| | 10 | 0.985359 | | 0.985360 | |
| | exact value of (3) or (4): 1.97071532 | | | | |
| | exact value of (5): 0.98535766 $\leftarrow$ | | | | |
| $n=-5$ | 1 | 1.010909 | | 1.011111 | |
| | 2 | 1.009891 | | 1.009798 | |
| | 3 | 1.010254 | $\leftarrow$ | 1.010331 | $\leftarrow$ |
| | 4 | 1.009940 | $\leftarrow$ | 1.009818 | $\leftarrow$ |
| | 5 | 1.010589 | | 1.011015 | |
| | 6 | 1.005300 | | 0.998322 | |
| | 7 | 0.951894 | | 0.887892 | |
| | 8 | 0.737202 | | 0.459630 | |
| | 9 | 0.134729 | | $-0.725230$ | |
| | 10 | $-1.243041$ | | $-3.418810$ | |
| | exact value of (3) or (4): 1.01011438 $\leftarrow$ | | | | |
| | exact value of (5): 0.50505719 | | | | |

Table 4: Values of the right-hand sides of (3) or (4) for the parameters $a = 11.7$, $b = 11.2$, $c = 11.4$.